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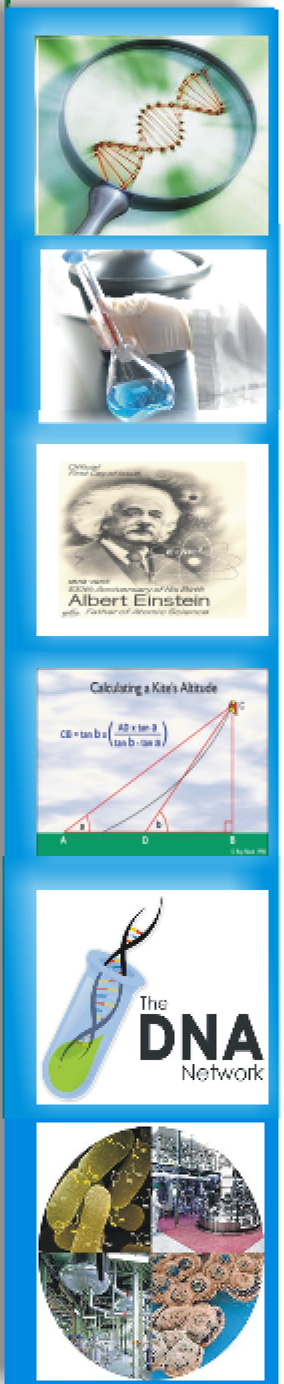


# CSIR NET - MATHEMATICAL SCIENCE

## SAMPLE THEORY

### SEQUENCES , SERIES AND LIMIT POINTS OF SEQUENCES

- *SEQUENCES*
- *LIMITS : INFERIOR & SUPERIOR*
- *ALGEBRA OF SEQUENCES*
- *SEQUENCE TESTS*
- *FOURIER SERIES*
- *SOME PROBLEMS*

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# SEQUENCE

A sequence in a set S is a function whose domain is the set N of natural numbers and whose range is a subset of S. A sequence whose range is a subset of R is called a real sequence.

$S_n = u_1 + u_2 + u_3 + ...... + u_n$

$S_1 = u_1$

$S_2 = u_1 + u_2$

$S_3 = u_1 + u_2 + u_3$

…

…

$S_n = u_1 + u_2 + u_3 + .... + u_n \rightarrow$ series

$\downarrow$

**Sequence**

**Bounded Sequence:** A sequence is said to be bounded if and only if its range is bounded. Thus a sequence $S_n$ is bounded if there exists

$$k \le S_n \le K, \ \forall n \in N$$

$$\Leftrightarrow S_n \in [k, K]$$

The l. u. b (Supremum) and the g.l.b (infimum) of the range of a bounded sequence may be referred as its g.l.b and l.u.b respectively.

## Limits inferior and Superior

From the definition of limit in Section 1.4, it follows that the limiting behavior of any sequence $\{a_n\}$ of real numbers, depends only on sets of the form $\{a_n : n \ge m\}$, i.e., $\{a_m, a_{m+1}, a_{m+2}, . . . \}$. In this regard we make the following definition.

**Definition** Let $\{a_n\}$ be a sequence of real numbers (not necessarily bounded). We define

$$\lim_{n\to\infty} \inf a_n = \sup_n \inf \{a_n, a_{n+1}, a_{n+2}, . . . \}$$

And $$\lim_{n\to\infty} \sup a_n = \inf_n \sup \{a_n, a_{n+1}, a_{n+2}, . . . \}$$

As the limit inferior and limit superior respectively of the sequence $\{a_n\}$.

We shall denote limit inferior and limit superior of $\{a_n\}$ by $\underline{\lim}_{n\to\infty} a_n$ and $\overline{\lim}_{n\to\infty} a_n$ or simply by $\underline{\lim}\, a_n$ and $\overline{\lim}\, a_n$ respectively.

We shall use the following notations for the sequence $\{a_n\}$, for each $n \in N$

$$\underline{A}_n = \inf \{a_n, a_{n+1}, a_{n+2}, \ldots\},$$

And $$\bar{A}_n = \sup \{a_n, a_{n+1}, a_{n+2}, \ldots\}.$$

Therefore, we have

$$\underline{\lim}\, a_n = \sup_n \underline{A}_n$$

And $$\overline{\lim}\, a_n = \inf_n \bar{A}_n$$

Now $\{a_{n+1}, a_{n+2}, \ldots\} \subseteq \{a_n, a_{n+1}, a_{n+2}, \ldots\}$, Therefore by taking infimum and supremum respectively, it follows that

$$\underline{A}_{n+1} \geq \underline{A}_n \text{ And } \bar{A}_{n+1} \leq \bar{A}_n$$

This is true for each $n \in \mathbf{N}$.

The above inequalities show that the associated sequences $\{\underline{A}_n\}$ and $\{\bar{A}_n\}$ monotonically increase and decrease respectively with n.

**Remark:** It should be noted that both limits inferior and superior exist uniquely (finite or infinite) for all real sequences.

**Theorem:** If $\{a_n\}$ is any sequence, then

$$\underline{\lim}\,(-a_n) = -\overline{\lim}, \text{ and } \overline{\lim}\,(-a_n) = -\underline{\lim}\, a_n.$$

Let $b_n = -a_n$, $n \in N$ then we have

$$\underline{B}_n = \inf \{b_n, b_{n+1}, \ldots\}$$

$$= -\sup \{a_n, a_{n+1}, \ldots\} = -\bar{A}_n$$

And so

$$\underline{\lim}\,(-a_n)\ \underline{\lim} = b_n = \sup(\underline{B}_1, \underline{B}_2, \ldots\ldots)$$

$$= \sup\{-\bar{A}_1, -\bar{A}_2, \ldots\}$$

$$= -\inf\{\bar{A}_1, \bar{A}_2, \ldots\}$$

$$= -\inf \bar{A}_n = -\overline{\lim}\, a_n.$$

Also,

$$\underline{\lim}\, a_n = \underline{\lim}\,(-(a_n)) = -\overline{\lim}\,(-a_n).$$

**Theorem:** If $\{a_n\}$ is any sequence, then

$\underline{\lim}\, a_n = -\infty$ if and only if $\{a_n\}$ is not bounded below, And $\overline{\lim}\, a_n = +\infty$ if and only if $\{a_n\}$ is not bounded above.

Let $\underline{A}_n = \inf\{a_n, a_{n+1}, \ldots\}$,

And $\bar{A}_n = \sup\{a_n, a_{n+1}, \ldots\}, n \in N$

By definition we have

$$\underline{\lim}\, a_n = -\infty \Leftrightarrow \sup\{\underline{A}_1, \underline{A}_2, \ldots\} = -\infty$$

$\Leftrightarrow$ $\underline{A}_n = -\infty, \quad \forall\ n \in \mathbf{N}$

$\Leftrightarrow$ $\inf\{a_n, a_{n+1}, \ldots\} = -\infty, \ \forall\ n \in \mathbf{N}$

$\Leftrightarrow$ $\{a_n\}$ is not bounded below:

The proof for limit superior is similar.

**Corollary:** If $\{a_n\}$ is any sequence, then

(i) $-\infty < \underline{\lim}\, a_n \leq +\infty$ iff $\{a_n\}$ is bounded below.

and

(ii) $-\infty \leq \overline{\lim}\, a_n < +\infty$ iff $\{a_n\}$ is bounded above.

For bounded sequences, we have the following useful criteria for limits inferior and superior respectively.

## Limit pts of a sequence.

A number $\xi$ is said to be a limit point of a sequence $S_n$ if given any nbd of $\xi$, $S_n$ belongs to the same for an infinite number of values of n.

Now $\{S_{n+1}\ S_{n+2}, S_{n+3}, \ldots\} \subseteq \{S_n, S_{n+1}, S_{n+2}, \ldots\}$, therefore by taking infimum and supremum respectively, if follows that $\underline{A}_{n+1} \geq \underline{A}_n$ and $\overline{A}_{n+1} \leq \overline{A}_n$ for each $n \in N$

**Remark:** Both limits inferior and superior exist uniquely (finite or infinite) for all real sequence.

**Theorem:** If $\{S_n\}$ is any sequence, then

$$\inf S_n \leq \underline{\lim}\, S_n \leq \operatorname{Sup} S_n$$

If $\{S_n\}$ is any sequence, then

$$\underline{\lim}\{-S_n\} = -\overline{\lim}\, S_n$$

And $-\overline{\lim}\{-S_n\} = \underline{\lim}\, S_n$

## Some Important Properties of Algebra of sequences

**1.** If $\{a_n\}$ is a bounded sequence such that $a_n > 0$ for all $n \in N$, then

(i) $\underline{\lim}\left(\frac{1}{a_n}\right) = \frac{1}{\overline{\lim} a_n}$, if $\overline{\lim} a_n > 0$

(ii) $\underline{\lim}\left(\frac{1}{a_n}\right) = \frac{1}{\underline{\lim}\, a_n}$, if $\underline{\lim} a_n > 0$

**5.** If $\{a_n\}$ and $\{b_n\}$ are bounded sequence, $a_n \geq 0, b_n > 0$ for all $n \in N$, then

(i) $\underline{\lim}\left(\frac{a_n}{b_n}\right) \geq \frac{\underline{\lim} a_n}{\overline{\lim}\, b_n}$, if $\overline{\lim}\, b_n > 0$

(ii) $\overline{\lim}\left(\frac{a_n}{b_n}\right) \leq \frac{\overline{\lim} a_n}{\underline{\lim}\, b_n}$, if $\underline{\lim}\, b_n > 0$

## SOME IMPORTANT SEQUENCE TESTS

### 1. Cauchy's root test

Let $\Sigma u_n$ be +ve term series and

$$\lim_{n\to\infty} \{u_n\}^{u_n} = \ell$$

Then the series is

(i) Cgt if $\ell < 1$

(ii) Dgt if $\ell > 1$

(iii) No firm decision is possible if $\ell = 1$

### 2. Raabe's test

Let $\Sigma u_n$ be a +ve term series and

$$\lim n\left\{\frac{u_n}{u_{n+1}} - 1\right\} = \ell$$

then the series is

(i) Cgt if $\ell > 1$

(ii) Dgt if $\ell < 1$

(iii) No firm decision is possible if $\ell = 1$

### 3. Logarithmic Test:

If $\Sigma u_n$ is +ve terms series such that

$$\lim_{n\to\infty}\left(n \log \frac{u_n}{u_{n+1}}\right) = \ell$$

Then the series

(i) cgt if $\ell > 1$

(ii) dgt if $\ell < 1$

## 4. Absolute convergent

A series $\Sigma u_n$ is said to be absolutely cgt if the positive term series $\Sigma|u_n|$ formed by the moduli of the terms of the series is convergent.

## 5. Conditional convergent

A series is said to be conditionally convergent if it is convergent without being absolutely convergent.

**Theorem:** Every absolute convergent series is convergent.

**Note. (i)** If $\Sigma u_n$ is cgt without being absolutely cgt. I.e. if $\Sigma u_n$ is conditionally cgt then each of the +ve term series $\Sigma g(n)$ and $\Sigma h(n)$ diverges to infinity which follows from

$$g(n) = \frac{1}{2}\left[|u_n| + u_n\right]$$

$$h(n) = \frac{1}{2}\left[|u_n| - u_n\right]$$

**(ii)** It should be noted that three are no comparison tests for the cgt of conditionally cgt series.

**Alternating series**

A series whose terms are alternately +ve and –ve is called an alternating series

## 6. Leibnitz's test

Let u be a sequence such that $\forall n \in N$

(i) $u_n \geq 0$

(ii) $u_{n+1} \leq u_n$

(iii) lim u = 0

Then alternating series u(1) – u(2) + u(3) – u(4) + .... + $(-1)^{n+1}$ u(n) ..... is cgt.

## 7. Abel's Test

If $a_n$ is a positive, monotonic decreasing function and if $\Sigma u_n$ is convergent series, then the series $\Sigma u_n a_n$ is also convergent.

## Uniform convergence

**Point wise Convergence of Sequence of Functions**

**Definition:** A sequence of functions $\{f_n\}$ defined on [a, b] is said to be point-wise convergent to a function f on [a, b], if

to each $\in > 0$ to each $x \in [a, b]$, there exists a positive integer m (depending on $\varepsilon$ and the point x) such that

$$|f_n(x) - f(x)| < \varepsilon \ \forall \ n > m \text{ and } \forall \ x \in [a,b].$$

The function f is called the point-wise limit of the sequence $\{f_n\}$. We write $\lim_{n\to\infty} f_n(x) = f(x)$.

## FOURIER SERIES

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\alpha} a_n \cos nx + \sum_{n=1}^{n} b_n \sin nx$$

Where $(0 < x < 2\pi)$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx \, dx$$

And $$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx \, dx$$

And for $(-\pi < x < \pi)$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

And $$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

Where f(x) is an odd function; $a_0 = 0$ and $a_n = 0$ where f(x) is an even function; $b_n = 0$.

Fourier series in the interval $(0 < x < 2\ell)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

Where $a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

And $$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

In the interval $(-\ell < x < \ell)$

$$a_0 = \frac{1}{l}\int_{-l}^{+l} f(x)\,dx,\quad a_n = \frac{1}{l}\int_{-l}^{+l} f(x)\cos\frac{n\pi x}{l}\,dx$$

And $b_n = \frac{1}{l}\int_{-l}^{+l} f(x)\sin\frac{n\pi x}{l}\,dx$

**Note:** When f(x) is an odd function, $a_0 = 0$ and $a_n = 0$ when f(x) is an even function, $b_n = 0$.

### Half-Range series $(0 < x < \pi)$

A function f(x) defined in the interval $0 < x < \pi$ has two distinct half-range series.

(i) The half-range cosine series is

$$f(x) = \frac{a_0}{2} + \sum a_n \cos nx$$

Where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$ and $a_n = \int_0^{\pi} f(x)\cos nx\,dx$

(ii) The half range sine series is,

$$f(x) = \Sigma b_n \sin nx$$

Where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx.$

### Half-Range Series $(0 < x < l)$

A function f (x) defined in the interval $(0 < x < l)$ and having two distinct half-range series.

(i) The half range cosine series is,

$$f(x) = \frac{a_0}{2} + \Sigma a_n \cos\frac{n\pi x}{l}$$

Where $a_0 = \frac{2}{l}\int_0^{l} f(x)\,dx$

And $$a_n = \frac{2}{l}\int_0^{l} f(x)\frac{\cos n\pi x}{l}\,dx$$

(ii) The half-range sine series is,

$$f(x) = \Sigma b_n \sin \frac{n\pi x}{l}$$

Where $b_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l} dx$

### Complex form of Fourier Series

$$f(x) = \sum_{m=-\infty}^{+\infty} c_m e^{imx}$$

Where $c_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-imx}dx$

$c_0 = \int_{-\pi}^{+\pi} f(x)dx$ and

$C_{-m} = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(x)e^{imx} dx.$

### Parseval's Identity

For Fourier series,

$$f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}+\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}, 0<x<2l$$

The Parseval's identity is

$$\frac{1}{2l}\int_0^{2l} \left[f(x)\right]^2 dx=\frac{a_0^2}{4}+\frac{1}{2}\sum_{n=1}^{\infty} \left(a_n^2 +b_n^2\right)$$

### FOURIER INTEGRAL

The Fourier series of periodic function f (x) on the interval $(-\ell,+\ell)$ is given by

$$f(x) = a_0 + \frac{n\pi x}{\ell} \cos \frac{n\pi x}{\ell} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell} \quad .....(1)$$

Where $a_0 = \frac{1}{2\ell}\int_{-\ell}^{+\ell} f(x)\,dx = \frac{1}{2\ell}\int_{-\ell}^{+\ell} f(t)\,dt$

$a_n = \frac{1}{\ell}\int_{-\ell}^{+\ell} f(t)\cos\frac{n\pi t}{\ell} dt$

$b_n = \frac{1}{\ell}\int_{-\ell}^{+\ell} f(t)\sin\frac{n\pi t}{\ell} dt$

Then

$$f(x) = \frac{1}{\pi}\int_0^{\infty} du \int_{-\infty}^{+\infty} f(t)\cos u(x-t)\,dt$$

This is a form of Fourier Integral.

## SOME PROBLEMS

**1.** The set of all positive values of a for which the series $\sum_{n=1}^{\infty}\left(\frac{1}{n}-\tan^{-1}\left(\frac{1}{n}\right)\right)^a$ converges, is

(1) $\left(0,\frac{1}{3}\right)$ (2) $\left(0,\frac{1}{3}\right)$ (3) $\left(\frac{1}{3},\infty\right)$ (4) $\left(\frac{1}{3},\infty\right)$

**2.** Match the following

| Series (X) | Domain of convergence (Y) |
|---|---|
| A. $\sum\frac{x^n}{n^3}$ | (i) [0, 2] |
| B. $\sum(-1)^n\frac{x^{2n+1}}{2n+1}$ | (ii) [–2 –e, –2 + e] |
| C. $\sum\frac{(-1)^{n+1}}{n}(x-1)^n$ | (iii) [–1, 1] |
| D. $\sum\frac{n!(x+2)^n}{n^n}$ | (iv) [–1, 1] |

| | A | B | C | D |
|---|---|---|---|---|
| (1) | (iv) | (iii) | (ii) | (i) |
| (2) | (iv) | (iii) | (i) | (ii) |
| (3) | (iii) | (iv) | (i) | (ii) |
| (4) | (i) | (ii) | (iv) | (iii) |

**3.** The series

$$1^p + \left(\frac{1}{2}\right)^p + \left(\frac{1.3}{2.4}\right)^p + \left(\frac{1.3.5}{2.4.6}\right)^p + \ldots \text{ is –}$$

(1) Convergent, if $p \geq 2$ divergent, if $p < 2$

(2) Convergent, if $p > 2$ and divergent, if $p \leq 2$

(3) Convergent, if $p \leq 2$ and divergent, if $p > 2$

(4) Convergent, if $p < 2$ and divergent, if $p \geq 2$

**4.** For the improper integral $\int_0^1 x^{\alpha-1}e^{-x}dx$ which one of the following is true

(1) if $\alpha < 0$, convergent and if $\alpha = 0$, divergent

(2) if $\alpha \geq 0$, Convergent and if $\alpha < 0$, divergent

(3) if $\alpha > 0$, convergent and if $\alpha \leq 0$, divergent

(4) If $\alpha > 0$, divergent and if $\alpha \leq 0$, convergent

**5.** Let $A \subseteq R$ and Let $f_1\, f_2 - f_n$ be functions on A to R and Let c be a cluster point of A if $L_k = \lim_{x\to c} f_k$ for k = 1, ....., n Then $\lim_{x\to c} [f(x)]^c$

(1) L (2) $L_k\; k \in N$ (3) $L^n$ (4) 1

***ANSWER KEY: - 1. (4), 2. (2), 3. (2), 4. (3), 5. (3)***

**1. (4) Use the following results:**

(1) Let $\Sigma a_n$ & $\Sigma b_n$ be two positive term series

(i) If $\underset{n\to\infty}{\text{Lt}} \frac{a_n}{b_n} = \ell$, $\ell$ being a finite non–zero constant, then $\Sigma a_n$ & $\Sigma b_n$ both converge or diverge together.

(ii) If $\underset{n\to\infty}{\text{Lt}} \frac{a_n}{b_n} = 0$ & $\Sigma\beta\nu$ χονϖεργεσ, then $\Sigma a_n$ also converges.

(2) The series $\sum \frac{1}{n^p}$ converges if $p > 1$ & diverges if $p \leq 1$. We compare the given series with the series $\sum \frac{1}{n^{ap}}$

$$\underset{n\to\infty}{\text{Lt}} \frac{\left(\frac{1}{n} - \tan^{-1}\frac{1}{n}\right)^a}{\frac{1}{n^{ap}}} = \underset{n\to\infty}{\text{Lt}} \frac{\left(\frac{1}{3n^3} - \frac{1}{5n^5}.......\right)^a}{\frac{1}{n^{pa}}} \left[\because \frac{1}{n} - \tan^{-1}\left(\frac{1}{n}\right) = \frac{1}{n} - \left[\frac{1}{n} - \frac{1}{3n^3} + ........\right]\right]$$

$$= \frac{1}{3n^3} - \frac{1}{5n^5} + ........$$

$$= \underset{n\to\infty}{\text{Lt}} \left(\frac{n^p}{3n^3} - \frac{n^p}{5n^5} - - - -\right)^a$$

For this limit to be zero or some other finite number

$3 - p \geq 0$ i.e. $p \leq 3$

& for the series $\sum \frac{1}{n^{ap}}$ to be convergent, ap > 1

$\Rightarrow \quad a > \frac{1}{p} \geq \frac{1}{3}$

$\Rightarrow \quad a > \frac{1}{3}$

$\Rightarrow \quad a \in \left(\frac{1}{3}, \infty\right)$ $\therefore$ **Ans.** is **(D)**

**2. (2)** **(i)** $\sum \frac{x^n}{n^3}$

$\because a_n = \frac{1}{n^3}; a_{n+1} = \frac{1}{(n+1)^3}$

$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^3 = 1$

So the domain of $a_n$ is [–1, 1] $\sum \frac{1}{n^2}$

For x = 1 the given power series is

Which is convergent.

For x = –1 the given power series is

$-1 + \frac{1}{2^3} - \frac{1}{3^3} + \frac{1}{4^3}...$

Which is convergent, by leibnitz's test. $\therefore$ **Ans.** is (iv)

(ii) $\sum (-1)^n \frac{x^{2n+1}}{2n+1}$

$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| == \lim_{n\to\infty}\frac{2n+3}{2n+1} = 1$

The interval of convergence [–1, 1]

for x = 1, the series becomes

$1 - \frac{1}{3} + \frac{1}{5}...$ Which is convergent by Leibnitz's test

For x = –1 the series becomes $-1 + \frac{1}{3} - \frac{1}{5}...$

Which is again convergent.

Hence the exact interval of convergency is [–1, 1]. $\therefore$ **Ans.** is (iii)

(iii) $R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{n}{n-1}\right| = 1$

Since the given power series is about the point x = 1 the interval of convergence is

–1 + 1 < x < 1 + 1 = 0 < x < 2

for x = +2, the given series $\sum \frac{(-1)^{n+1}}{n}$ which is convergent by leibnitz's test.

Hence the exact interval of convergence is [ 0, 2]. $\therefore$ **Ans.** is (i)

(iv) $\sum \frac{n!(x+2)^n}{n^n}$

The given power series is about the point x = 2

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\frac{n!}{n^n}\cdot\frac{(n+1)^{n+1}}{(n+1)!}$$

$$= \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$ $\therefore$ **Ans.** is (ii)

The interval of convergence is [–2 –e, –2+ e],

**3. (2)** Neglecting the first term

$$u_n = \left(\frac{1.3.5.....(2n-1)}{2.4.6.....2n}\right)^p$$

and $u_{n+1} = \left(\frac{1.3.5.....(2n-1)(2n+1)}{2.4.6.....(2n)(2n+2)}\right)^p$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \left(\frac{2n+2}{2n+1}\right)^p = \frac{\left(1+\frac{1}{n}\right)^p}{\left(1+\frac{1}{2n}\right)^p}$$

or, $\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)^p}{\left(1+\frac{1}{2n}\right)^p} = 1$

$\therefore$ Ratio test fails.

$$\therefore \log \frac{u_n}{u_{n+1}} = \log \left\{ \frac{\left(1+\frac{1}{n}\right)^p}{\left(1+\frac{1}{2n}\right)^p} \right\}$$

$$= p\log\left(1+\frac{1}{n}\right) - p\log\left(1+\frac{1}{2n}\right)$$

$$= p\left[\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-...\right)-\left(\frac{1}{2n}-\frac{1}{8n^2}+\frac{1}{24n^3}\right)\right]$$

$$= p\left[\left(\frac{1}{n}-\frac{1}{2n^2}\right)-\left(\frac{1}{2n}-\frac{1}{8n^2}\right)+\left(\frac{1}{3n^3}-\frac{1}{24n^3}\right)+...\right]$$

$$= p\left[\frac{1}{2n}-\frac{3}{8n^2}+\frac{7}{24n^3}+....\right]$$

$$\therefore \lim_{n\to\infty} n\log\frac{u_n}{u_{n+1}}$$

$$= \lim_{n\to\infty} p\left(\frac{1}{2}-\frac{3}{8n}+\frac{7}{24n^2}+....\right)$$

$$= \frac{p}{2}$$

From Logarithmic test.

The series is convergent, if $\frac{1}{2}p > 1$, i.e., $p > 2$

The series is divergent, if $\frac{1}{2}p < 1$, i.e., $p < 2$

The test fails, if $\frac{1}{2}p = 1$ i.e., $p = 2$

Now $n \log \frac{u_n}{u_{n+1}} = 2\left(\frac{1}{2}-\frac{3}{8n}+\frac{7}{24n^2}+...\right)$

or, $\left\{n\log\frac{u_n}{u_{n+1}}-1\right\}$

$$= \left\{\left(1-\frac{3}{4n}+\frac{7}{12n^2}+...\right)-1\right\}$$

$$= -\frac{3}{4n} + \frac{7}{12n^2} + \ldots\ldots$$

or, $\left\{n\log\frac{u_n}{u_{n+1}} - 1\right\}\log n$

$$= -\frac{3}{4} \times \frac{\log n}{n} + \frac{7}{12} \times \frac{\log n}{n^2} + \ldots.$$

or, $\lim_{n\to\infty}\left(-\frac{3}{4}\times\frac{\log n}{n}+\frac{7}{12}\times\frac{\log n}{n^2}....\right)$

Hence by higher logarithmic test the given series is divergent, if p = 2.

Hence the given series is convergent when p > 2 and divergent when $p \le 2$.

The correct answer is (2).

**4. (3)** $\int_0^1 x^{\alpha-1}e^{-x}dx,$

When $\alpha$ > 1, the given integral is a proper integral and hence it is convergent. When $\alpha$ < 1, the integrand becomes infinite at x = 0.

Now $\lim_{x\to 0} x^{\mu}.x^{\alpha-1}e^{-x} = \lim_{x\to 0} x^{\mu+\alpha-1}e^{-x} = 1$

if $\mu+\alpha-1=0$, i.e., $\mu = 1-\alpha$

We then have 0 < $\mu$ < 1 when 0 < $\alpha$ < 1

and $\mu \geq 1$ where $\alpha \leq 0$.

It follows by $\mu$ -test that the integral is convergent when 0 < $\alpha$ < 1 and divergent when $\alpha \leq 0$.

And we have proved above that the integral is convergent when $\alpha \geq 1$. Consequently the given integral is convergent if $\alpha$ > 0 and divergent if $\alpha \leq 0$.

**5. (3)** if $L_k = \lim_{x\to c} f_k$

then it follows from a by known result which is called an Induction arument that

$$L_1 + L_2 + \cdots + L_n = \lim_{x\to c} f(_1 + f_2 + \cdots + f_n),$$

and

$$L_1 \cdot L_2 \cdots L_n = \lim(f_1 \cdot f_2 \cdots f_n).$$

In particular, we deduce that if L = $\lim_{x\to c}$ f and n $\in$ N, then

$$L^n = \lim_{x\to c} (f(x))^n.$$